

STABILIZATION OF THE NONLINEAR DAMPED WAVE EQUATION VIA LINEAR WEAK OBSERVABILITY

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ABSTRACT. We consider the problem of energy decay rates for nonlinearly damped abstract infinite dimensional systems. We prove sharp, simple and quasi-optimal energy decay rates through an indirect method, namely a weak observability estimate for the corresponding undamped system. One of the main advantage of these results is that they allow to combine the optimal-weight convexity method of [3, Alabau-Boussouira] and a methodology of [6, Ammari-Tucsnak] for weak stabilization by observability. Our results extend to nonlinearly damped systems, those of Ammari and Tucsnak [6]. At the end, we give an appendix on the weak stabilization of linear evolution systems.

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1. INTRODUCTION

We consider the following second order differential equation

$$(1.1) \quad \begin{cases} \ddot{w}(t) + Aw(t) + a(\cdot)\rho(\cdot, \dot{w}) = 0, & t \in (0, \infty), x \in \Omega \\ w(0) = w^0, \dot{w}(0) = w^1. \end{cases}$$

where Ω is a bounded open set in \mathbb{R}^N , with a boundary Γ and $\rho : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is supposed to be continuous on $\overline{\Omega} \times \mathbb{R}$ and strictly monotone with respect to the second variable. We assume that Ω is either convex or of class $\mathcal{C}^{1,1}$. We set

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$H = L^2(\Omega)$, with its usual scalar product denoted by $\langle \cdot, \cdot \rangle_H$ and the associated norm $\| \cdot \|_H$ and where $A : D(A) \subset H \rightarrow H$ is a densely defined self-adjoint linear operator satisfying

$$(1.2) \quad \langle Au, u \rangle_H \geq C \|u\|_H^2 \quad \forall u \in D(A)$$

for some $C > 0$. We also introduce the scale of Hilbert spaces H_α , as follows: for every $\alpha \geq 0$, $H_\alpha = \mathcal{D}(A^\alpha)$, with the norm $\|z\|_\alpha = \|A^\alpha z\|_H$. The space $H_{-\alpha}$, is defined by duality with respect to the pivot space H as follows: $H_{-\alpha} = H_\alpha^*$, for $\alpha > 0$. The operator A can be extended (or restricted) to each H_α , such that it becomes a bounded operator

$$(1.3) \quad A : H_\alpha \rightarrow H_{\alpha-1} \quad \forall \alpha \in \mathbb{R}.$$

Assumption (A1): There exists a continuous strictly increasing odd function $g \in \mathcal{C}([-1, 1]; \mathbb{R})$, continuously differentiable in a neighbourhood of 0 and satisfying $g(0) = g'(0) = 0$, with

$$(1.4) \quad \begin{cases} c_1 g(|v|) \leq |\rho(\cdot, v)| \leq c_2 g^{-1}(|v|), & |v| \leq 1, \text{ a.e. on } \Omega, \\ c_1 |v| \leq |\rho(\cdot, v)| \leq c_2 |v|, & |v| \geq 1, \text{ a.e. on } \Omega, \end{cases}$$

where g^{-1} denotes the inverse function of g and $c_i > 0$ for $i = 1, 2$. Moreover $a \in \mathcal{C}(\bar{\Omega})$, with $a \geq 0$ on Ω and there exists $a_0 > 0$ such that $a(x) \geq a_0$ on ω . Here ω stands for the subregion of Ω on which the feedback ρ is active.

The equation (1.1) is understood as an equation in $H_{-1/2}$, i.e., all the terms are in $H_{-1/2}$. The energy of a solution is defined by

$$(1.5) \quad E_w(t) = \frac{1}{2} \left(\|(w(t), \dot{w}(t))\|_{H_{1/2} \times H}^2 \right)$$

Most of the nonlinear equations modelling the damped vibrations of elastic structures can be written in the form (1.1), where w stands for the displacement field and the term $B\dot{w}(t) = a(\cdot)\rho(\cdot, \dot{w})$, represents a viscous feedback damping.

Let us introduce the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & -a\rho \end{pmatrix} : D(\mathcal{A}) = H_1 \times H_{1/2} \subset H_{1/2} \times H \rightarrow H_{1/2} \times H$$

and (1.1) becomes

$$\dot{W} = \mathcal{A}W, \quad W(0) = W^0,$$

where $W^0 = \begin{pmatrix} w^0 \\ w^1 \end{pmatrix}$ and $W = \begin{pmatrix} w \\ \dot{w} \end{pmatrix}$.

The operator \mathcal{A} is the generator of a continuous semigroup of nonlinear contractions in $H_{1/2} \times H$ (see [8, Corollary 2.1, page 35]). Then the system (1.1) is well-posed. More precisely, the following holds:

If $(w^0, w^1) \in H_1 \times H_{1/2}$. Then the problem (1.1) admits a unique strong solution

$$w \in C([0, \infty); H_1) \cap C^1([0, \infty); H_{1/2}).$$

Moreover, if $(w^0, w^1) \in H_{1/2} \times H$ then the system (1.1) admits a unique mild solution, i.e., $(w, \dot{w}) \in C([0, +\infty), H_{1/2} \times H)$.

We have for all $t \geq 0$, the following energy identity:

(1.6)

$$\|(w^0, w^1)\|_{H_{1/2} \times H}^2 - \|(w(t), \dot{w}(t))\|_{H_{1/2} \times H}^2 = 2 \int_0^t \int_{\Omega} a(\cdot) \rho(\cdot, \dot{w}(s)), \dot{w}(s) dx ds.$$

The aim of this paper is to deduce energy decay rates from weak observability estimates for the associated undamped system, that is

$$(1.7) \quad \begin{cases} \ddot{\phi}(t) + A\phi(t) = 0, \\ \phi(0) = \phi^0, \dot{\phi}(0) = \phi^1. \end{cases}$$

Our results extend to nonlinearly damped systems, those of Ammari and Tucsnak [6] (see also [7] for more details) which concern linearly damped systems.

2. PRELIMINARIES AND MAIN RESULTS

Before stating our main results, let us precise some hypotheses on the feedback and give some preliminary definitions.

We define a function R (see [3]) by

$$(2.1) \quad R(x) = \sqrt{x}g(\sqrt{x}), \quad x \in [0, r_0^2],$$

Thanks to assumption **(A1)**, R is of class \mathcal{C}^1 and is strictly convex on $[0, r_0^2]$, where $r_0 > 0$ is a sufficiently small number. We still denote by R its extension to \mathbb{R} with $R(x) = +\infty$ for $x \in \mathbb{R} \setminus [0, r_0^2]$. We also define a function L by

$$(2.2) \quad L(y) = \begin{cases} \frac{R^*(y)}{y}, & \text{if } y \in (0, +\infty), \\ 0, & \text{if } y = 0, \end{cases}$$

where R^* stands for the convex conjugate function of R , i.e.: $R^*(y) = \sup_{x \in \mathbb{R}} \{xy - R(x)\}$. Moreover we define a weight function f such that

$$(2.3) \quad R^*(f(s)) = \frac{sf(s)}{\beta}, \quad s \in [0, \beta r_0^2],$$

where β is a constant that will be chosen later. We recall that f is defined by

$$f(s) = L^{-1}\left(\frac{s}{\beta}\right), \quad \forall s \in [0, \beta r_0^2].$$

One can show [3] that f is a strictly increasing function from $[0, \beta r_0^2)$ onto $[0, \infty)$.

After, we consider the unbounded operator

$$(2.4) \quad \mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \subset H_{1/2} \times H \rightarrow H_{1/2} \times H, \quad \mathcal{A}_d = \begin{pmatrix} 0 & I \\ -A & -a \end{pmatrix},$$

where

$$\mathcal{D}(\mathcal{A}_d) = H_1 \times H_{1/2}.$$

Let X_1, X_2 be two Banach spaces such that

$$\mathcal{D}(\mathcal{A}_d) \subset H_{1/2} \times H \subset X_1 \times X_2,$$

with continuous embeddings and

$$(2.5) \quad [H_1 \times H_{1/2}, X_1 \times X_2]_{\theta} = H_{1/2} \times H,$$

for a fixed real number $0 < \theta < 1$, where $[\cdot, \cdot]_\theta$ denotes the interpolation space (see for instance Triebel [15], [6]) and $\mathcal{G} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an increasing and continuous function on $\mathbb{R}_+ = (0, \infty)$.

Assumption (A2): There exist $T, C_T > 0$ such that the following observability inequality is satisfied for the linear conservative system (1.7)

$$(2.6) \quad c_T E_\phi(0) \mathcal{G} \left(\frac{\|(\phi^0, \phi^1)\|_{X_1 \times X_2}^2}{E_\phi(0)} \right) \leq \int_0^T |\sqrt{a} \dot{\phi}|_H^2 dt$$

for any non-identically zero initial data $(\phi^0, \phi^1) \in H_{1/2} \times H$.

Our main results are stated as follows:

Theorem 2.1. *Let $\eta > 0$ and $T_0 > 0$ be fixed given real numbers. For any $r \in (0, \eta)$, we define a function K_r from $(0, r)$ on $[0, \infty)$ by*

$$(2.7) \quad K_r(\tau) = \int_\tau^r \frac{1}{v(f\mathcal{G}_\theta)^{-1}(v)} dv,$$

here $\mathcal{G}_\theta = \mathcal{G} \circ x^{\frac{1}{\theta}-1}$. We also define

$$(2.8) \quad \psi_r(z) = z + K_r(f\mathcal{G}_\theta(\frac{1}{z})), \quad z \geq \frac{1}{(f\mathcal{G}_\theta)^{-1}(r)}.$$

Assume (A1) and (A2). Then for non-identically zero initial data $(w^0, w^1) \in H_1 \times H_{1/2}$, the energy of the strong solution of (1.1) satisfies

$$(2.9) \quad E_w(t) \leq \beta T (f\mathcal{G}_\theta)^{-1} \left(\frac{1}{\psi_r^{-1}(\frac{t-T}{T_0})} \right), \quad \text{for } t \text{ sufficiently large.}$$

Remark 2.2. Suppose further that the function

$$\begin{aligned} h : (0, 1) &\rightarrow \mathbb{R}_+ \\ x &\mapsto \frac{1}{x^{\frac{\theta}{1-\theta}}} \mathcal{G} \end{aligned}$$

is increasing on $(0, 1)$.

Notice that

$$h(\alpha x) \leq h(x), \forall \alpha \in (0, 1), x \in (0, 1),$$

or equivalently

$$\mathcal{G}(\alpha x) \leq \alpha^{\frac{\theta}{1-\theta}} \mathcal{G}(x), \forall \alpha \in (0, 1), x \in (0, 1).$$

Letting α goes to zero this implies that $\mathcal{G}(0) = 0$ and then $\mathcal{G}(x) > 0$ for all $x > 0$. In this case the inequality (2.6) implies, according to [6, Theorem 2.4], that we have a weak stability for the linear associated problem, i.e., there exists a constant $C > 0$ such that for all $t > 0$ and for all $(w^0, w^1) \in H_1 \times H_{\frac{1}{2}}$ we have that the solution of (1.1) with $\rho = Id$ satisfies:

$$E_w(t) \leq C \left[\mathcal{G}^{-1} \left(\frac{1}{1+t} \right) \right]^{\frac{\theta}{1-\theta}} \| (w^0, w^1) \|_{H_1 \times H_{\frac{1}{2}}}^2.$$

Let $\mathcal{H} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that \mathcal{H} is continuous, invertible and increasing on \mathbb{R}_+ .

Assumption (A3): There exist $T, C_T > 0$ such that the following observability inequality is satisfied for the linear conservative system (1.7)

$$(2.10) \quad C_T \|(\phi^0, \phi^1)\|_{H_1 \times H_{\frac{1}{2}}}^2 \mathcal{H} \left(\frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|_{H_1 \times H_{\frac{1}{2}}}^2} \right) \leq \int_0^T |\sqrt{a}\dot{\phi}|_H^2 dt$$

for any non-identically zero initial data $(\phi^0, \phi^1) \in H_1 \times H_{\frac{1}{2}}$.

By the same way as in Theorem 2.7 we have the following result.

Theorem 2.3. *Let $\eta > 0$ and $T_0 > 0$ be fixed given real numbers. For any $r \in (0, \eta)$, we define a function \mathcal{K}_r from $(0, r)$ on $[0, \infty)$ by*

$$(2.11) \quad \mathcal{K}_r(\tau) = \int_\tau^r \frac{1}{v(f\mathcal{H})^{-1}(v)} dv.$$

We also define

$$(2.12) \quad \Psi_r(z) = z + \mathcal{K}_r(f\mathcal{H}(\frac{1}{z})), \quad z \geq \frac{1}{(f\mathcal{H})^{-1}(r)}.$$

Assume (A1) and (A3). Then for non-identically zero initial data $(w^0, w^1) \in H_1 \times H_{1/2}$, the energy of the strong solution of (1.1) satisfies

$$(2.13) \quad E_w(t) \leq \beta T (f\mathcal{H})^{-1} \left(\frac{1}{\Psi_r^{-1}(\frac{t-T}{T_0})} \right), \quad \text{for } t \text{ sufficiently large.}$$

Remark 2.4. (1) If we suppose in addition that the function $x \mapsto \frac{1}{x} \mathcal{H}(x)$ is increasing on $(0, 1)$. Then, the estimate (2.13) is a generalization (to the nonlinear case) of (6.7) in Theorem 6.1.

(2) The case $\mathcal{H} = \text{Id}$ corresponds to the situation treated in [4, Theorem 1.1] (which we can compare to the linear case, i.e., Theorem 6.1.)

3. INTERMEDIATE RESULTS

We start by a key Lemma which relies on the optimal-weight convexity method of [3] (see also [4, 1, 2]), so the proof will be omitted.

Lemma 3.1. *Assume that ρ and a satisfy the assumption (A1) and that there exists $r_0 > 0$ sufficiently small so that the function R defined by (2.1) is strictly convex on $[0, r_0^2]$. Let $(w^0, w^1) \in H_1 \times H_{1/2}$, non-identically zero, be given and $(\phi^0, \phi^1) = (w^0, w^1)$ and w and ϕ be the respective solutions of (1.1) and of (1.7). Then the following inequality holds*

$$(3.1) \quad \begin{aligned} & \int_0^T f \left(\frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2} \right) \int_\Omega \left(a(x)|\dot{w}|^2 + a(x)|\rho(x, \dot{w})|^2 \right) dx dt \\ & \leq c_5 T R^* \left(f \left(\frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2} \right) \right) \\ & + c_6 \left(f \left(\frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2} \right) + 1 \right) \int_0^T \int_\Omega a(x) \rho(x, \dot{w}) \dot{w} dx dt, \end{aligned}$$

where

$$c_5 = |\Omega|(1 + c_2^2), \quad c_6 = \left(\frac{1}{c_1} + c_2\right),$$

and $|\Omega| = \int_{\Omega} d\sigma$, with $d\sigma = a(\cdot)dx$.

The next Lemma compares the localized kinetic damping of the linearly damped equation with the localized linear and nonlinear kinetic energies of the nonlinearly damped equation.

Lemma 3.2. *Assume that $\rho \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}; \mathbb{R})$ is a continuous monotone nondecreasing function with respect to the second variable on Ω such that $\rho(\cdot, 0) = 0$ on Ω . Let w be the solution of (1.1) with non-identically zero initial data $(w^0, w^1) \in H_1 \times H_{1/2}$. Let us introduce z solution of the linear locally damped problem*

$$(3.2) \quad \begin{cases} \ddot{z} + Az + a(x)\dot{z} = 0, \\ z(0) = w^0, \dot{z}(0) = w^1. \end{cases}$$

Then the following inequality holds

$$(3.3) \quad \int_0^T \int_{\Omega} a(x)|\dot{z}|^2 dx dt \leq 2 \int_0^T \int_{\Omega} \left(a(x)|\dot{w}|^2 + a(x)|\rho(x, \dot{w})|^2 \right) dx dt.$$

The next Lemma compares the localized observation for the conservative undamped equation with the localized damping of the linearly damped equation.

Lemma 3.3. *Assume that $a \in \mathcal{C}(\overline{\Omega})$, with $a \geq 0$ on Ω . Let $T > 0$ be given, then there exists $k_T > 0$ (given by $k_T = 8T^2 \|a\|_{L^\infty(\Omega)}^2 + 2$) such that for all $(w^0, w^1) \in H_1 \times H_{1/2}$*

$$(3.4) \quad \int_0^T \int_{\Omega} a|\dot{\phi}|^2 dx dt \leq k_T \int_0^T \int_{\Omega} a|\dot{z}|^2 dx dt$$

where ϕ is the solution of the conservative equation (1.7) with $(\phi^0, \phi^1) = (w^0, w^1)$ and z is the solution of (3.2).

4. PROOF OF THE MAIN RESULTS

The following lemmas will be very useful.

Lemma 4.1. *Let $\delta > 0$ and M be an increasing and a non-negative function such that the function defined by $\psi(x) = x - \rho_T M(x)$ is strictly increasing on $[0, \delta]$, for some positive constant ρ_T . Assume that \widehat{E} is a nonnegative, nonincreasing function defined on $[0, \infty)$ with $\widehat{E}(0) < \delta$ and satisfying*

$$(4.1) \quad \widehat{E}((k+1)T) \leq \widehat{E}(kT) - \rho_T M(\widehat{E}(kT)), \quad \forall k \in \mathbb{N}.$$

After we consider the sequence $(\widetilde{y}_k)_k$ defined by induction as follows:

$$(4.2) \quad \begin{cases} \widetilde{y}_{k+1} - \widetilde{y}_k + \rho_T M(\widetilde{y}_k) = 0, \quad k \in \mathbb{N}, \\ \widetilde{y}_0 = E_0. \end{cases}$$

Then the following inequality holds

$$(4.3) \quad E_k \leq \widetilde{y}_k,$$

here we set

$$(4.4) \quad E_k = \widehat{E}(kT), \quad \forall k \in \mathbb{N}.$$

Proof. Since the sequence $(\tilde{y}_k)_k$ satisfies (4.2), so we have

$$(4.5) \quad E_{k+1} - \widetilde{y_{k+1}} \leq \psi(E_k) - \psi(\tilde{y}_k), \quad \forall k \in \mathbb{N}.$$

We prove (4.3) par induction on k . Since $E_0 \leq \tilde{y}_0$, (4.3) holds for $k = 0$. Assume that (4.3) holds at the order k . First, we remark that since \hat{E} is nonincreasing and thanks to our assumption $E_0 < \delta$, we have

$$E_k < \delta, \quad \forall k \in \mathbb{N}.$$

Moreover, it is easy to check that the sequence $(\tilde{y}_k)_k$ is nonincreasing, so that

$$\tilde{y}_k \leq \tilde{y}_0 = E_0 < \delta, \quad \forall k \in \mathbb{N}.$$

Thanks to our choice of δ , and since we make the assumption that $E_k \leq \tilde{y}_k$, we deduce that

$$\psi(E_k) - \psi(\tilde{y}_k) \leq 0.$$

Using this last estimate in (4.5), we deduce that (4.3) holds at the order $k + 1$. ■

We now compare the sequence (\tilde{y}_k) obtained using an Euler scheme to the solution of the associated ordinary differential equation at time kT .

Lemma 4.2. *Assume the hypotheses of Lemma 4.1. We define E_k as in (4.4). We consider the ordinary differential equation*

$$(4.6) \quad \begin{cases} y'(s) + \frac{\rho T}{T} M(y(s)) = 0, & s \geq 0, \\ y(0) = E_0 \end{cases}$$

and set

$$(4.7) \quad s_k = kT, y_k = y(s_k), \quad \forall k \in \mathbb{N}.$$

Then we have for all k in \mathbb{N}

$$(4.8) \quad \tilde{y}_k \leq y_k,$$

where $(\tilde{y}_k)_k$ is defined by (4.2).

Proof. We integrate (4.6) between s_k and s_{k+1} and compare with the equation satisfied by \tilde{y}_k . Thus we have

$$(4.9) \quad y_{k+1} - \widetilde{y_{k+1}} - (y_k - \tilde{y}_k) + \frac{\rho T}{T} \int_{s_k}^{s_{k+1}} (M(y(s)) - M(\tilde{y}_k)) ds = 0, \quad \forall k \in \mathbb{N}.$$

We prove (4.8) by induction on k . The property clearly holds for $k = 0$. Assume that it holds at the order k . Since y is nonincreasing, we deduce that $y_k = y(s_k) \leq y_0 = E_0 < \delta$. Thus

$$y(s) \leq y_k < \delta, \quad \forall s \in [s_k, s_{k+1}].$$

Since M is nondecreasing, we deduce from (4.9) that

$$\left(\psi(y_k) - \psi(\tilde{y}_k) \right) \leq y_{k+1} - \widetilde{y_{k+1}}.$$

Since we assume that (4.8) holds at the order k and since ψ is nondecreasing on $[0, \delta]$, we deduce

$$0 \leq \left(\psi(y_k) - \psi(\tilde{y}_k) \right).$$

Using this last inequality in the above one, we prove (4.8) at the order $k + 1$. ■

We deduce from Lemmas 4.1 and 4.2 the following result.

Corollary 4.3. *Assume the hypotheses of Lemma 4.1 and Lemma 4.2. Then we have*

$$(4.10) \quad E_k \leq y(s_k), \quad \forall k \in \mathbb{N}.$$

The proof of the main result rely on the following abstract theorem of which proof based on the previous lemmas is given in [4].

Theorem 4.4. *Let $\eta > 0$ and $T_0 > 0$ be fixed given real numbers and let F be strictly increasing function from $[0, +\infty)$ onto $[0, \eta)$, with $F(0) = 0$ and $\lim_{y \rightarrow \infty} F(y) = \eta$. For any $r \in (0, \eta)$, we define a function K_r from $(0, r)$ on $[0, \infty)$ by*

$$(4.11) \quad K_r(\tau) = \int_{\tau}^r \frac{1}{vF^{-1}(v)} dv$$

We also define

$$(4.12) \quad \psi_r(z) = z + K_r\left(F\left(\frac{1}{z}\right)\right), \quad z \geq \frac{1}{F^{-1}(r)}.$$

Let $T > 0$ and $\rho_T > 0$ be given. Let $\delta > 0$ be such that the function defined by $x \mapsto x - \rho_T x F^{-1}(x)$ is strictly increasing on $[0, \delta]$. Assume that \widehat{E} is a nonnegative, nonincreasing function defined on $[0, \infty)$ with $\widehat{E}(0) < \delta$ and satisfying

$$(4.13) \quad \widehat{E}((k+1)T) \leq \widehat{E}(kT) \left(1 - \rho_T F^{-1}(\widehat{E}(kT))\right), \quad \forall k \in \mathbb{N}.$$

Then \widehat{E} satisfies the upper estimate

$$(4.14) \quad \widehat{E}(t) \leq TF\left(\frac{1}{\psi_r^{-1}\left(\frac{(t-T)\rho_T}{T_0}\right)}\right), \quad \text{for } t \text{ sufficiently large.}$$

We repeat the proof for the reader's convenience.

Proof of Theorem 4.4. We set

$$(4.15) \quad T_0 = \frac{T}{\rho_T}, \quad r = \widehat{E}(0), \quad M(v) = vF^{-1}(v).$$

Thus the solution y of (4.6) is characterized as

$$(4.16) \quad y(t) = K_r^{-1}\left(\frac{t}{T_0}\right), \quad t \geq 0.$$

On the other hand, we define E_k by (4.4). Then, thanks to (4.13), E_k satisfies

$$(4.17) \quad E_{k+1} \leq E_k \left(1 - \rho_T F^{-1}(E_k)\right), \quad \forall k \in \mathbb{N}.$$

Let $l \in \mathbb{N}$ be an arbitrary fixed integer. We have in particular

$$E_{k+1+i} - E_{k+i} + \rho_T M(E_{k+i}) \leq 0, \quad \text{for } i = 0 \dots, i = l.$$

Summing these inequalities from $i = 0$ to $i = l$, and using the fact that $(E_k)_k$ is a nonincreasing sequence whereas M is a nondecreasing function, we obtain

$$E_{k+l+1} - E_k + \frac{1}{T_0}(l+1)TM(E_{k+l}) \leq 0$$

so that

$$(4.18) \quad (l+1)TM(E_{k+l}) \leq T_0 E_k, \quad \forall k, l \in \mathbb{N}.$$

In particular, we have for any arbitrary $p \in \mathbb{N}$

$$(4.19) \quad M(E_p) \leq \frac{T_0}{T} \inf_{l \in \{0, \dots, p\}} \left(\frac{E_{p-l}}{l+1} \right).$$

Now thanks to Corollary 4.3 and to (4.16), we have

$$E_i \leq y_i = K_r^{-1} \left(\frac{iT}{T_0} \right), \quad \forall i \in \mathbb{N}.$$

Using this last relation in (4.19), we deduce that

$$(4.20) \quad M(E_p) \leq \frac{T_0}{T} \inf_{l \in \{0, \dots, p\}} \left(\frac{K_r^{-1} \left(\frac{(p-l)T}{T_0} \right)}{l+1} \right).$$

Let now $t \geq T$ be given and $p \in \mathbb{N}$ be the unique integer so that $t \in [pT, (p+1)T)$. Let $\theta \in (0, t - T]$ be arbitrary and $l \in \mathbb{N}$ be the unique integer so that $\theta \in [lT, (l+1)T)$. Then, thanks to (4.20) and by construction, we have

$$M(\widehat{E}(t)) \leq M(E_p) \leq \frac{T_0}{T} \inf_{l \in \{0, \dots, p\}} \left(\frac{K_r^{-1} \left(\frac{(p-l)T}{T_0} \right)}{l+1} \right),$$

and

$$K_r^{-1} \left(\frac{(p-l)T}{T_0} \right) \leq K_r^{-1} \left(\frac{t - \theta - T}{T_0} \right).$$

We deduce that

$$M(\widehat{E}(t)) \leq \frac{T}{\theta} K_r^{-1} \left(\frac{t - T - \theta}{T_0} \right), \quad \forall \theta \in (0, t - T].$$

Using the fact that M is strictly increasing, we obtain

$$\widehat{E}(t) \leq TM^{-1} \left(\inf_{\theta \in (0, (t-T)]} \left(\frac{1}{\theta} K_r^{-1} \left(\frac{t - T - \theta}{T_0} \right) \right) \right).$$

Let now $t > 0$ be fixed for the moment and put $\gamma_t(\theta) = \frac{1}{\theta} K_r^{-1} \left(\frac{t - T - \theta}{T_0} \right)$. Thus θ^* is a critical point of γ_t if and only if it satisfies the relation:

$$K_r^{-1} \left(\frac{t - T - \theta^*}{T_0} \right) + \frac{\theta^*}{T_0 K_r' K_r^{-1} \left(\frac{t - T - \theta^*}{T_0} \right)} = 0.$$

Hence θ^* is a critical point of γ_t if and only if it solves the equation

$$K_r^{-1} \left(\frac{t - T - \theta^*}{T_0} \right) = \frac{\theta^*}{T_0} M \left(K_r^{-1} \left(\frac{t - T - \theta^*}{T_0} \right) \right).$$

Using the definition of M , we deduce that θ^* is a critical point of γ_t if and only if it satisfies the following equation:

$$\frac{T_0}{\theta^*} = F^{-1} \left(K_r^{-1} \left(\frac{t - T - \theta^*}{T_0} \right) \right)$$

Hence θ^* is a critical point of γ_t if and only if it verifies the following equation:

$$\psi_r \left(\frac{\theta^*}{T_0} \right) = \frac{t - T}{T_0},$$

and we obtain

$$\widehat{E}(t) \leq TF \left(\frac{1}{\psi_r^{-1} \left(\frac{t - T}{T_0} \right)} \right), \quad \forall t \geq T.$$

So that (4.14) is proved. \blacksquare

Proof of Theorem 2.1.

$$\begin{aligned}
& \int_0^T f \left(\frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2} \right) \int_\Omega \left(a(x)|\dot{w}|^2 + a(x)|\rho(x, \dot{w})|^2 \right) dx dt \\
& \geq c_T f \left(\frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2} \right) \int_0^T \int_\Omega a(x)|\dot{\phi}|^2 dx dt \\
& \geq c_T f \left(\frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2} \right) \|(\phi^0, \phi^1)\|_{H_{1/2} \times H}^2 \mathcal{G} \left(\frac{\|(\phi^0, \phi^1)\|_{X_1 \times X_2}^2}{\|(\phi^0, \phi^1)\|_{H_{1/2} \times H}^2} \right) \\
& \geq c_T f \left(\frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2} \right) \|(\phi^0, \phi^1)\|_{H_{1/2} \times H}^2 \mathcal{G} \left(\left(\frac{\|(\phi^0, \phi^1)\|_{H_{1/2} \times H}^2}{\|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2} \right)^{\frac{1}{\theta}-1} \right),
\end{aligned}$$

here $c_T = \frac{1}{2k_T}$.

Since

$$R^* \left(f \left(\frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2} \right) \right) = \frac{E_\phi(0)}{\beta \|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2} f \left(\frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2} \right),$$

this together with (3.1) and the definition of the weight function f lead to:
(4.21)

$$C_T E_\phi(0) f(\widehat{E}_w(0)) \mathcal{G}((\widehat{E}_w(0))^{\frac{1}{\theta}-1}) \leq \frac{C_T}{\beta} \widehat{E}_w(0) f(\widehat{E}_w(0)) + C_7(E_w(0) - E_w(T)),$$

where we put $\widehat{E}_w(0) = \frac{E_\phi(0)}{\|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2}$. Moreover,

$$(4.22) \quad C_T \widehat{E}_w(0) f(\widehat{E}_w(0)) \mathcal{G}((\widehat{E}_w(0))^{\frac{1}{\theta}-1}) \leq C_T \frac{\widehat{E}_w(0) f(\widehat{E}_w(0))}{\beta \|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2} + C_7(\widehat{E}_w(0) - \widehat{E}_w(T)).$$

gives

$$(4.23) \quad \widehat{E}_w(T) \leq \widehat{E}_w(0) [1 - (C'_T \mathcal{G}((\widehat{E}_w(0))^{\frac{1}{\theta}-1}) - \frac{C_8 T}{\beta \|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2}) f(\widehat{E}_w(0))].$$

Choose β so that $(C'_T \mathcal{G}((\widehat{E}_w(0))^{\frac{1}{\theta}-1}) - \frac{C_8 T}{\beta \|(\phi^0, \phi^1)\|_{H_1 \times H_{1/2}}^2}) > C''_T \mathcal{G}((\widehat{E}_w(0))^{\frac{1}{\theta}-1})$.

Hence

$$(4.24) \quad \widehat{E}_w(T) \leq \widehat{E}_w(0) [1 - (C''_T \mathcal{G}((\widehat{E}_w(0))^{\frac{1}{\theta}-1})) f(\widehat{E}_w(0))].$$

Make use of Theorem 4.4, the proof is complete. \blacksquare

Proof of Theorem 2.3. The proof is a simple adaptation of the proof of Theorem 2.1. \blacksquare

5. SOME APPLICATIONS

We give applications of Theorems 2.1 and 2.3. In the next result, we denote by C a positive constant depending on $E(0)$ and T . Also, we give only the expression of g in a right neighbourhood of 0, since as long as g has a linear growth at infinity, the asymptotic behavior of the energy depends only on the behavior of g close to 0.

We assume that ρ and a satisfy assumption **(A1)**. We assume that there exists $T > 0$ such that the solution of (1.7) satisfies the weak observability inequality (2.6) for example 1 below and the assumption (2.10) for examples 2 and 3. Then, we have the following results:

5.1. Example 1. Let g be given by $g(x) = x^p$, $p > 1$ on $(0, r_0]$. Then the energy of solution of (1.1) satisfies the estimate

$$E_w(t) \leq C \left(x \mapsto x^{\frac{p-1}{2}} \mathcal{G}_\theta(x) \right)^{-1} \left(\frac{1}{t+1} \right),$$

for t sufficiently large and for all any non-identically zero initial data $(w^0, w^1) \in H_1 \times H_{1/2}$.

5.2. Example 2. Let g and \mathcal{H} are given by $g(x) = x^p$, $p > 1$ on $(0, r_0]$. Then the energy of solution of (1.1) satisfies the estimate

$$E_w(t) \leq \left(x \mapsto x^{\frac{p-1}{2}} \mathcal{H}(x) \right)^{-1} \left(\frac{1}{t+1} \right),$$

for t sufficiently large and for all any non-identically zero initial data $(w^0, w^1) \in H_1 \times H_{1/2}$.

Particular case: For $\mathcal{H}(x) = \exp\left(-\frac{C}{x^p}\right)$, $C, p > 0$ the last estimate becomes

$$E_w(t) \leq \frac{C}{(\ln(1+t))^p}.$$

5.3. Example 3. Let g be given by $g(x) = x^3 \exp\left(-\frac{1}{x^2}\right)$. Then the energy of solution of (1.1) satisfies the estimate

$$E_w(t) \leq C \left(x \mapsto \exp\left(-\frac{1}{x}\right) \mathcal{H}(x) \right)^{-1} \left(\frac{1}{1+t} \right),$$

for t sufficiently large and for all any non-identically zero initial data $(w^0, w^1) \in H_1 \times H_{1/2}$.

Particular cases: For $\mathcal{H}(x) = \exp\left(-\frac{C}{x^p}\right)$, $C, p > 0$ the last estimate becomes :

$$E_w(t) \leq \frac{C}{\ln(1+t)}, \text{ for } p \geq 1,$$

and

$$E_w(t) \leq \frac{C}{(\ln(1+t))^p}, \text{ for } p < 1.$$

5.4. **Example 4.** Here we consider the following initial and boundary problem:

$$(5.1) \quad \begin{cases} u_{tt} - \Delta u + a(x)\rho(x, u_t) = 0, & (x, t) \in \Omega \times (0, +\infty), \\ u = 0, & \text{on } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), & \text{on } \Omega, \end{cases}$$

where ρ and a satisfy assumption **(A1)** and Ω is a convex bounded open set of \mathbb{R}^N of class \mathcal{C}^2 .

In this case, we have:

$$A = -\Delta : D(A) \subset H = L^2(\Omega) \rightarrow L^2(\Omega), \quad H_1 = D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

$H_{1/2} = H_0^1(\Omega)$ and A is a selfadjoint operator satisfying (1.2).

Moreover the conservative equation (1.7) becomes in this case:

$$(5.2) \quad \begin{cases} \phi_{tt} - \Delta \phi = 0, & \Omega \times (0, +\infty), \\ \phi = 0, & \partial\Omega \times (0, +\infty), \\ \phi(x, 0) = \phi^0(x), \quad \phi_t(x, 0) = \phi^1(x), & \Omega. \end{cases}$$

According to [13] we show that the observability inequality is given by

Proposition 5.1. *For all $\beta \in]0, 1[$ there exists T and $c_T > 0$ such that the following observability inequality holds:*

$$(5.3) \quad \begin{aligned} & \|(\phi^0, \phi^1)\|_{[H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)}^2 \exp \left[-c_T \left(\frac{\|(\phi^0, \phi^1)\|_{[H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)}}{\|(\phi^0, \phi^1)\|_{H_0^1(\Omega) \times L^2(\Omega)}} \right)^{1/\beta} \right] \\ & \leq \int_0^T |\sqrt{a}\dot{\phi}|_H^2 dt, \end{aligned}$$

for all non-identically zero initial data $(\phi^0, \phi^1) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$.

We remark here that we have (2.10) for $\mathcal{H}(x) = \exp(-\frac{c_T}{x^{1/2\beta}})$, $\forall x > 0$.

Thus according to Theorem 2.3 we have the following stabilization result for the nonlinear damped wave equation as in [12, 10, 9].

Theorem 5.2. *We suppose that $\text{meas}(\text{supp } a) \neq 0$. Then, the energy of solution of (5.1) satisfies for all $\beta \in]0, 1[$ the estimate:*

$$(5.4) \quad E_w(t) \leq \frac{C}{(\ln(1+t))^{2\beta}}, \quad \text{for } t \text{ sufficiently large}$$

and for all any non-identically zero initial data $(u^0, u^1) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$.

6. APPENDIX: WEAK STABILIZATION OF LINEAR EVOLUTION SYSTEMS

Let H be a Hilbert space with the norm $\|\cdot\|_H$, and let $A : \mathcal{D}(A) \subset H \rightarrow H$ be a self-adjoint, positive and boundedly invertible operator. We also introduce the scale of Hilbert spaces H_α , as follows: for every $\alpha \geq 0$, $H_\alpha = \mathcal{D}(A^\alpha)$, with the norm $\|z\|_\alpha = \|A^\alpha z\|_H$. The space $H_{-\alpha}$, is defined by duality with respect to the pivot space H as follows: $H_{-\alpha} = H_\alpha^*$, for $\alpha > 0$.

Let the bounded linear operator $B : U \rightarrow H$, where U is another Hilbert space which will be identified with its dual.

The system we consider is described by

$$(6.1) \quad \ddot{w}(t) + Aw(t) + BB^*\dot{w}(t) = 0, w(0) = w_0, \dot{w}(0) = w_1, t \in [0, \infty),$$

The system (6.1) is well-posed:

For $(w_0, w_1) \in H_{\frac{1}{2}} \times H$, the problem (6.1) admet a unique solution

$$w \in C([0, \infty); H_{\frac{1}{2}} \times H)$$

such that $B^*\dot{w}(\cdot) \in L_{loc}^2(0, +\infty; U)$. Moreover, w satisfies the energy estimate, for all $t \geq 0$

$$(6.2) \quad \|(w_0, w_1)\|_{H_{\frac{1}{2}} \times H}^2 - \|(w(t), \dot{w}(t))\|_{H_{\frac{1}{2}} \times H}^2 = 2 \int_0^t \|B^*\dot{w}(s)\|_U^2 ds.$$

For (6.2) we remark that the mapping $t \mapsto \|(w(t), \dot{w}(t))\|_{H_{\frac{1}{2}} \times H}^2$ is non-increasing.

Consider the initial value problem :

$$(6.3) \quad \ddot{\varphi}(t) + A\varphi(t) = 0,$$

$$(6.4) \quad \varphi(0) = \varphi_0, \dot{\varphi}(0) = \varphi_1.$$

It is well known that (6.3)-(6.4) is well posed in $H_1 \times H_{\frac{1}{2}}$ and in $H_{\frac{1}{2}} \times H$.

Now, we consider the unbounded linear operator

$$(6.5) \quad \mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \subset H_{\frac{1}{2}} \times H \rightarrow H_{\frac{1}{2}} \times H, \mathcal{A}_d = \begin{pmatrix} I & 0 \\ -A & -BB^* \end{pmatrix},$$

where

$$\mathcal{D}(\mathcal{A}_d) = H_1 \times H_{\frac{1}{2}}.$$

Let $\mathcal{H} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that \mathcal{H} is continuous, invertible, increasing on \mathbb{R}_+ and suppose that the function $x \mapsto \frac{1}{x} \mathcal{H}(x)$ is increasing on $(0, 1)$.

In the case of non exponential decay in the energy space we have the explicit decay estimate valid for regular initial data, which is a simple adaptation of [6, Theorem 2.4].

Theorem 6.1. *Assume that the function \mathcal{H} satisfies the assumptions above. Then the following assertion holds true:*

If for all non-identically zero initial data $(\varphi_0, \varphi_1) \in H_1 \times H_{\frac{1}{2}}$ we have

$$(6.6) \quad \int_0^T \|B^*\dot{\varphi}(t)\|_U^2 dt \geq C \|(\varphi_0, \varphi_1)\|_{H_1 \times H_{\frac{1}{2}}}^2 \mathcal{H} \left(\frac{\|(\varphi_0, \varphi_1)\|_{H_{\frac{1}{2}} \times H}^2}{\|(\varphi_0, \varphi_1)\|_{H_1 \times H_{\frac{1}{2}}}^2} \right),$$

for some constant $C > 0$ then there exists a constant $C_1 > 0$ such that for all $t > 0$ and for all non-identically zero initial data $(w_0, w_1) \in H_1 \times H_{\frac{1}{2}}$ we have

$$(6.7) \quad \|(w(t), \dot{w}(t))\|_{H_{\frac{1}{2}} \times H}^2 \leq C_1 \mathcal{H}^{-1} \left(\frac{1}{1+t} \right) \|(w_0, w_1)\|_{H_1 \times H_{\frac{1}{2}}}^2.$$

Remark 6.2. *In the case where $\mathcal{H} = Id$ the observability inequality (6.6) is equivalent to the exponential stability of (6.1), see [6, Theorem 2.2].*

Proof. We suppose (6.6), which implies that there exist $C, T > 0$ such that for all non-identically zero initial data $(w^0, w^1) \in H_1 \times H_{\frac{1}{2}}$ we have

$$\int_0^T \|B^* \dot{\varphi}(t)\|_U^2 dt \geq C \|(w_0, w_1)\|_{H_{\frac{1}{2}} \times H}^2 \mathcal{H} \left(\frac{\|(w_0, w_1)\|_{H_{\frac{1}{2}} \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H_{\frac{1}{2}}}^2} \right).$$

By applying [6, Lemma 4.1] we obtain that the solution $w(t)$ of (6.1) satisfies the following inequality

$$\int_0^T \|B^* \dot{w}(t)\|_U^2 dt \geq C \|(w_0, w_1)\|_{H_{\frac{1}{2}} \times H}^2 \mathcal{H} \left(\frac{\|(w_0, w_1)\|_{H_{\frac{1}{2}} \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H_{\frac{1}{2}}}^2} \right).$$

Relation above and (6.2) imply the existence of a constant $K > 0$ such that

$$\begin{aligned} \|(w(T), \dot{w}(T))\|_{H_{\frac{1}{2}} \times H}^2 &\leq \|(w_0, w_1)\|_{H_{\frac{1}{2}} \times H}^2 \\ (6.8) \quad &- K \|(w_0, w_1)\|_{H_{\frac{1}{2}} \times H}^2 \mathcal{H} \left(\frac{\|(w_0, w_1)\|_{H_{\frac{1}{2}} \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H_{\frac{1}{2}}}^2} \right). \end{aligned}$$

By using the fact that the function $t \mapsto \|(w(t), \dot{w}(t))\|_{H_{\frac{1}{2}} \times H}^2$ is nonincreasing, the function \mathcal{H} is increasing and relation (6.8) we obtain the existence of a constant $K_1 > 0$ such that

$$\begin{aligned} \|(w(T), \dot{w}(T))\|_{H_{\frac{1}{2}} \times H}^2 &\leq \|(w_0, w_1)\|_{H_{\frac{1}{2}} \times H}^2 \\ (6.9) \quad &- K_1 \|(w_0, w_1)\|_{H_{\frac{1}{2}} \times H}^2 \mathcal{H} \left(\frac{\|(w(T), \dot{w}(T))\|_{H_{\frac{1}{2}} \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H_{\frac{1}{2}}}^2} \right). \end{aligned}$$

Estimate (6.9) remains valid in successive intervals $[kT, (k+1)T]$, so, we have

$$\begin{aligned} \|(w((k+1)T), \dot{w}((k+1)T))\|_{H_{\frac{1}{2}} \times H}^2 &\leq \|(w(kT), \dot{w}(kT))\|_{H_{\frac{1}{2}} \times H}^2 \\ &- K_1 \|(w(kT), \dot{w}(kT))\|_{H_{\frac{1}{2}} \times H}^2 \mathcal{H} \left(\frac{\|(w((k+1)T), \dot{w}((k+1)T))\|_{H_{\frac{1}{2}} \times H}^2}{\|(w(kT), \dot{w}(kT))\|_{H_1 \times H_{\frac{1}{2}}}^2} \right). \end{aligned}$$

Since \mathcal{A}_d generates a semigroup of contractions in $\mathcal{D}(\mathcal{A}_d)$, relations above imply the existence of a constant $K_2 > 0$ such that

$$\begin{aligned} \|(w((k+1)T), \dot{w}((k+1)T))\|_{H_{\frac{1}{2}} \times H}^2 &\leq \|(w(kT), \dot{w}(kT))\|_{H_{\frac{1}{2}} \times H}^2 \\ (6.10) \quad &- K_2 \|(w(kT), \dot{w}(kT))\|_{H_{\frac{1}{2}} \times H}^2 \mathcal{H} \left(\frac{\|(w((k+1)T), \dot{w}((k+1)T))\|_{H_{\frac{1}{2}} \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H_{\frac{1}{2}}}^2} \right), \end{aligned}$$

If we adopt now the notation

$$(6.11) \quad \mathcal{E}_k = \mathcal{H} \left(\frac{\|(w(kT), \dot{w}(kT))\|_{H_{\frac{1}{2}} \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H_{\frac{1}{2}}}^2} \right),$$

the inequality (6.10) implies

$$(6.12) \quad \frac{\|(w((k+1)T), \dot{w}((k+1)T))\|_{H_{\frac{1}{2}} \times H}^2}{\|(w(kT), \dot{w}(kT))\|_{H_{\frac{1}{2}} \times H}^2} \frac{\mathcal{E}_k}{\mathcal{E}_{k+1}} \mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}_k \mathcal{E}_{k+1}.$$

Since, the function $t \rightarrow \|(w(t), \dot{w}(t))\|_{H_{\frac{1}{2}} \times H}^2$ is nonincreasing and the function \mathcal{H} is increasing, relation (6.12) implies

$$(6.13) \quad \frac{\|(w((k+1)T), \dot{w}((k+1)T))\|_{H_{\frac{1}{2}} \times H}^2}{\|(w(kT), \dot{w}(kT))\|_{H_{\frac{1}{2}} \times H}^2} \frac{\mathcal{E}_k}{\mathcal{E}_{k+1}} \mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}_{k+1}^2.$$

According to (6.11), relation (6.13) gives,

$$(6.14) \quad \frac{\frac{1}{\|(w(kT), \dot{w}(kT))\|_{H_{\frac{1}{2}} \times H}^2} \mathcal{H} \left(\frac{\|(w(kT), \dot{w}(kT))\|_{H_{\frac{1}{2}} \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H_{\frac{1}{2}}}^2} \right)}{\frac{1}{\|(w_0, w_1)\|_{H_1 \times H_{\frac{1}{2}}}^2} \mathcal{H} \left(\frac{\|(w(kT), \dot{w}(kT))\|_{H_{\frac{1}{2}} \times H}^2}{\|(w_0, w_1)\|_{H_1 \times H_{\frac{1}{2}}}^2} \right)} \mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}_{k+1}^2.$$

Relation (6.14) combined with that the function $x \mapsto \frac{1}{x} \mathcal{H}(x)$ is increasing in $(0, 1)$, gives

$$(6.15) \quad \mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}_{k+1}^2, \quad \forall k \geq 0.$$

By applying [5, Lemma 5.2] and using relation (6.11) we obtain the existence of a constant $M > 0$ such that

$$\|(w(kT), \dot{w}(kT))\|_{H_{\frac{1}{2}} \times H}^2 \leq \mathcal{H}^{-1} \left(\frac{M}{k+1} \right) \|(w_0, w_1)\|_{H_1 \times H_{\frac{1}{2}}}^2, \quad \forall k \geq 0,$$

which obviously implies (6.7). ■

Example. We consider the following initial and boundary problem:

$$(6.16) \quad \begin{cases} u_{tt} - \Delta u + a(x) u_t = 0, & (x, t) \in \Omega \times (0, +\infty) \\ u = 0, & \text{on } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), & \text{on } \Omega, \end{cases}$$

where Ω is a convex bounded open set of \mathbb{R}^N of class \mathcal{C}^2 and $a \in \mathcal{C}(\overline{\Omega})$ with $a \geq 0$ on Ω and as in assumption **(A1)**.

In this case, we have:

$$A = -\Delta : \mathcal{D}(A) = H_1 \subset L^2(\Omega) \rightarrow L^2(\Omega), \quad H_1 = \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

$$H_{1/2} = H_0^1(\Omega), \quad U = L^2(\Omega) \text{ and } Bz = B^*z = \sqrt{a}z, \quad \forall z \in L^2(\Omega).$$

Moreover the conservative equation (1.7) becomes in this case:

$$(6.17) \quad \begin{cases} \phi_{tt} - \Delta \phi = 0, & \Omega \times (0, +\infty), \\ \phi = 0, & \partial\Omega \times (0, +\infty), \\ \phi(x, 0) = u^0(x), \phi_t(x, 0) = u^1(x), & \Omega. \end{cases}$$

According to [13] we show that the observability inequality is given by:

Proposition 6.3. *For all $\beta \in]0, 1[$ there exist $T, c_T > 0$ such that the following observability inequality holds:*

$$(6.18) \quad \begin{aligned} & \| (u^0, u^1) \|_{[H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)}^2 \exp \left[-c_T \left(\frac{\| (u^0, u^1) \|_{[H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)}}{\| (u^0, u^1) \|_{H_0^1(\Omega) \times L^2(\Omega)}} \right)^{1/\beta} \right] \\ & \leq \int_0^T \int_{\Omega} a(x) |\phi_t(x, t)|^2 dx dt, \end{aligned}$$

for all any non-identically zero initial data $(u^0, u^1) \in [H_0^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$.

We remark here that we have (6.6) for $\mathcal{H}(x) = \exp(-\frac{c_T}{x^{1/2\beta}})$, $\forall x > 0$. Thus according to Theorem 6.1 we have the following stabilization result for the linear wave equation which extends the result obtained by [11, Lebeau] (with a resolvent method).

Theorem 6.4. *For all $\beta \in]0, 1[$, there exists a constant $C > 0$ such that for all any non-identically zero initial data $(u^0, u^1) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$ the energy of the solution of (6.16) satisfies the estimate*

$$(6.19) \quad \| (u(t), \dot{u}(t)) \|_{H_0^1(\Omega) \times L^2(\Omega)} \leq \frac{C}{(\ln(1+t))^\beta} \| (u^0, u^1) \|_{[H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)}, \quad t > 0.$$

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